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On the Rationality of Algebraic Tori of Norm Type

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Let G be a finite group. A G-module is called a *permutation* module if it is isomorphic to $\bigoplus_{i=1}^{r} \mathbf{Z}G/G_i$, where G_i, $1 \leqq i \leqq r$, are subgroups of G. A G-module M is called a *quasi-permutation* module if there exists an exact sequence

$$0 \to M \to S \to S' \to 0,$$

where S and S' are permutation G-modules. The dual module $\mathrm{Hom}_Z(M, \mathbf{Z})$ of a G-module M is denoted by M°.

Let k be a field and let F be an extension field of k. Then F is said to be *rational* over k if it is generated by a finite number of elements of F which are algebraically independent over k and to be *stably rational* over k if there exists an extension field of F which is rational over each of k and F. Further, F is said to be *retract rational* over k if it is the quotient field of an integral domain $A \supseteq k$ such that there are k-algebra homomorphisms

$$\phi : A \to k[X_1, X_2, \dots, X_n][1/s] \qquad \text{and} \qquad \psi : k[X_1, X_2, \dots, X_n][1/s] \to A,$$

where $k[X_1, X_2, \dots, X_n][1/s]$ is the localization of a polynomial ring $k[X_1, X_2, \dots, X_n]$ with variables $X_1, X_2, \dots, X_n$ over k with respect to $0 \neq s \in k[X_1, X_2, \dots, X_n]$ and $\psi \cdot \phi$ is the identity on A. It is easy to see that *rational* $\Rightarrow$ *stably rational* $\Rightarrow$ *retract rational.*

Let G be a finite group and let M be a G-module with a $\mathbf{Z}$-free basis $u_1, u_2, \dots, u_n$. Let k be a field and let K be a Galois extension of k with group G. Define the action of G on the rational function field

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$K(X_1, X_2, \ldots, X_n)$ with variables $X_1, X_2, \ldots, X_n$ over K, as an extension of the action of G on K, as follows. For each $\sigma \in G$,

$$\sigma(X_i) = \prod_{j=1}^{n} X_j^{m_{ij}}, \qquad 1 \leqq i \leqq n,$$

when $\sigma u_i = \sum_{j=1}^{n} m_{ij} u_j, m_{ij} \in \mathbf{Z}$, and denote $K(X_1, X_2, \ldots, X_n)$ with this action of G by $K(M)$.

As is well known, there is an algebraic torus T defined over k and split over K such that the character group of T is isomorphic to M as G-modules, and the invariant subfield $K(M)^G$ of $K(M)$ can be identified with the function field of T over k.

We now have the following fundamental

PROPOSITION. *Let G be a finite group and let k be a field. Let K be a Galois extension of k with group G and let M be a $\mathbf{Z}$-free and $\mathbf{Z}$-finitely generated G-module* (*i.e., a G-lattice*). *Then*:

(1) (*e.g.*, [2, 1.6]) *M is a quasi-permutation G-module if and only if $K(M)^G$ is stably rational over k.*

(2) [7, 3.14] *M is a direct summand of a quasi-permutation G-module if and only if $K(M)^G$ is retract rational over k.*

Let p be a prime, and let P be an elementary abelian p-group of order p^m, $m \geqq 1$. Let P_i, $1 \leqq i \leqq r$, be distinct subgroups of index p in P, and let $\varepsilon_i : \mathbf{Z}P/P_i \to \mathbf{Z}$ be the augmentation maps. Further, for $h_1, h_2, \ldots, h_r \geqq 1$, let

$$\Phi = \left(\varepsilon_1^{h_1}, \varepsilon_2^{h_2}, \ldots, \varepsilon_r^{h_r}\right) : \bigoplus_{i=1}^{r} (\mathbf{Z}P/P_i)^{h_i} \to \mathbf{Z}$$

and set $L = \operatorname{Ker} \Phi$.

The main result of this note is the following

THEOREM 1. (1) *In case of $p = 2$, L° is a* (*direct summand of a*) *quasi-permutation P-module if and only if $r = 1, 2$.*

(2) *In case of $p \neq 2$, L° is a* (*direct summand of a*) *quasi-permutation P-module if and only if $r = 1$.*

By the Proposition, Theorem 1 can be restated as follows:

THEOREM 2. *Let G, P, and L be as above and let K be a Galois extension of a field k with group P. Then*:

(1) *In case of $p = 2$, $K(L^\circ)^G$ is stably* (*retract*) *rational over k if and only if $r = 1, 2$.*

(2) *In case of $p \neq 2$, $K(L^\circ)^G$ is stably (retract) rational over k if and only if $r = 1$.*

It is noted that the algebraic torus corresponding to L° defined over k and split over K is of *norm type*.

Part (1) of Theorem 2 is an answer to the question asked the late professor T. Miyata by A. Merkurjev in 1982 [6]. It should be noted that Theorem 1 was obtained in 1982 and reported without proof in [4].

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We now begin to prove Theorem 1. In order to prove this, we need to consider a more general situation. Let P, P_i, and ε_i, $1 \leqq i \leqq r$, be as above, and define the homomorphism $\delta : \mathbf{Z} \to \mathbf{Z}$ by $\delta(1) = p$. For $h_1, h_2, \ldots, h_r \geqq 1$ and $h \geqq 0$, let

$$\Phi = \left(\varepsilon_1^{h_1}, \varepsilon_2^{h_2}, \ldots, \varepsilon_r^{h_r}, \delta\right) : \bigoplus_{i=1}^{r} (\mathbf{Z}P/P_i)^{h_i} \oplus \mathbf{Z}^h \to \mathbf{Z},$$

and set $L = \operatorname{Ker} \Phi$. Further, let

$$\Phi' = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r) : \bigoplus_{i=1}^{r} \mathbf{Z}P/P_i \to \mathbf{Z},$$

and set $L' = \operatorname{Ker} \Phi'$. Then the key lemma is given as follows:

LEMMA. $L \cong L' \oplus \bigoplus_{i=1}^{r} (\mathbf{Z}P/P_i)^{h_i - 1} \oplus \mathbf{Z}^h$.

Proof. First, define the homomorphism $\nu_i : \mathbf{Z}P/P_i \to (\mathbf{Z}P/P_i)^{h_i}$ by $\nu_i(a_i) = (a_i, 0, 0, \ldots, 0)$ for each $1 \leqq i \leqq r$, and set

$$\nu = (\nu_1, \nu_2, \ldots, \nu_r) : \bigoplus_{i=1}^{r} \mathbf{Z}P/P_i \to \bigoplus_{i=1}^{r} (\mathbf{Z}P/P_i)^{h_i} \subseteq \bigoplus_{i=1}^{r} (\mathbf{Z}P/P_i)^{h_i} \oplus \mathbf{Z}^h.$$

Next, define the homomorphism $\mu_i : (\mathbf{Z}P/P_i)^{h_i} \to \mathbf{Z}P/P_i$ by $\mu_i((a_{i1}, a_{i2}, \ldots, a_{ih_i})) = \sum_{j=1}^{h_i} a_{ij}$ for each $1 \leqq i \leqq r$ and the homomorphism $\eta : \mathbf{Z}^h \to \bigoplus_{i=1}^{r} \mathbf{Z}P/P_i$ by $\eta((b_1, b_2, \ldots, b_h)) = ((\sum_{i=1}^{h} b_i s, 0, 0, \ldots, 0)$ ($\eta = 0$ when $h = 0$), where s denotes the sum of all elements of P/P_1 in $\mathbf{Z}P/P_1$, and set

$$\mu = (\mu_1, \mu_2, \ldots, \mu_r, \eta) : \bigoplus_{i=1}^{r} (\mathbf{Z}P/P_i)^{h_i} \oplus \mathbf{Z}^h \to \bigoplus_{i=1}^{r} \mathbf{Z}P/P_i.$$

Then $\mu \cdot \nu$ is the identity on $\bigoplus_{i=1}^{r} \mathbf{Z}P/P_i$ and it can be easily seen that $\nu(L') \subseteq L$ and $\mu(L) \subseteq L'$. Therefore, denoting the restrictions of ν, μ to L', L by $\tilde{\nu}$, $\tilde{\mu}$, respectively, we get homomorphisms $\tilde{\nu} : L' \to L$ and

$\tilde{\mu}: L \to L'$ such that $\tilde{\mu} \cdot \tilde{\nu}$ is the identity on L'. This shows that $L \cong L' \oplus \operatorname{Coker} \tilde{\nu}$. However, as is easily seen, $\operatorname{Coker} \tilde{\nu} \cong \operatorname{Coker} \nu \cong \bigoplus_{i=1}^r (\mathbf{Z}P/P_i)^{h_i - 1} \oplus \mathbf{Z}^h$. Thus this concludes that $L \cong L' \oplus \bigoplus_{i=1}^r (\mathbf{Z}P/P_i)^{h_i-1} \oplus \mathbf{Z}^h$.

By this lemma we have only to consider the case where $h_1 = h_2 = \cdots = h_r = 1$, that is, the exact sequence

$$0 \to L \to \bigoplus_{i=1}^{r} \mathbf{Z}P/P_i \to \mathbf{Z} \to 0, \qquad (*)$$

where $P_1, P_2, \ldots, P_r$ are distinct subgroups of index p in P.

It is known that L° is a quasi-permutation P-module for each of the cases (a) $r = 1$ and (b) $r = 2$ and $p = 2$ [6]. However, for completeness, we give a proof of it.

Assume first that $r = 1$. Then P/P_1 is cyclic, and so $L^\circ \cong L$. Hence L° is clearly a quasi-permutation P-module. Next, assume that $r = 2$ and $p = 2$. Then $P/P_1 \cap P_2$ is the Klein four group, and therefore, regarding $P/P_1 \cap P_2$ as P, we can set $P = \langle \sigma, \tau \mid \sigma^2 = \tau^2 = 1, \sigma\tau = \tau\sigma \rangle$ and $P_1 = \langle \sigma \rangle$, $P_2 = \langle \tau \rangle$. Now we have $\operatorname{Ker} \Phi = \mathbf{Z}P(1, -1)$. Define the epimorphism $\Psi = (\mu, \nu): \mathbf{Z}P \oplus \mathbf{Z} \to L$ by $\mu(1) = (1, -1)$ and $\nu(1) = (1 + \tau, 1 + \sigma)$. Then, as is easily seen, $\operatorname{Ker} \Psi = \{((a + b\sigma)(1 + \sigma\tau), -(a + b)) \mid a, b \in \mathbf{Z}\}$, and so $\operatorname{Ker} \Psi \cong \mathbf{Z}P/\langle \sigma\tau \rangle$. Thus we have the exact sequence $0 \to \mathbf{Z}P/\langle \sigma\tau \rangle \to \mathbf{Z}P \oplus \mathbf{Z} \to L \to 0$, which shows that L° is a quasi-permutation P-module.

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Consider here the case here $p = 2$ and $r \geqq 3$. This case can be divided into the following:

(1) $[P : P_{i_1} \cap P_{i_2} \cap P_{i_3}] = 4$ for some $i_1 < i_2 < i_3$.

(2) $[P : P_{i_1} \cap P_{i_2} \cap P_{i_3}] = 8$ for any $i_1 < i_2 < i_3$.

Set $P' = P_{i_1} \cap P_{i_2} \cap P_{i_3}$ in both cases and restrict the exact sequence $(*)$ to P'. Then we obtain the exact sequence

$$0 \to L^{P'} \to \mathbf{Z}P/P_{i_1} \oplus \mathbf{Z}P/P_{i_2} \oplus \mathbf{Z}P/P_{i_3} \oplus \bigoplus_{i \neq i_1, i_2, i_3} (\mathbf{Z}P/P_i)^{P'} \to \mathbf{Z} \to 0.$$

It is noted that L° is not (a direct summand of) a quasi-permutation P-module if $(L^{P'})^\circ$ is not (a direct summand of) a quasi-permutation P/P'-module. Regarding P/P' as P and applying the lemma to the above exact sequence, the cases (1) and (2) reduce, respectively, to the following

(i) $P = \langle \sigma, \tau \mid \sigma^2 = \tau^2 = 1, \sigma\tau = \tau\sigma \rangle$, the Klein four group,

$$0 \to L \to \mathbf{Z}P/\langle \sigma \rangle \oplus \mathbf{Z}P/\langle \tau \rangle \oplus \mathbf{Z}P/\langle \sigma\tau \rangle \to \mathbf{Z} \to 0.$$

(ii) $\mathrm{P} = \langle \rho, \sigma, \tau \mid \rho^2 = \sigma^2 = \tau^2 = 1,\ \rho\sigma = \sigma\rho,\ \sigma\tau = \tau\sigma,\ \tau\rho = \rho\tau \rangle$

(a) $0 \to L \to \mathbf{Z}P/\langle \rho, \sigma \rangle \oplus \mathbf{Z}P/\langle \tau, \rho \rangle \oplus \mathbf{Z}P/\langle \sigma, \tau \rangle \to \mathbf{Z} \to 0.$

(b) $0 \to L \to \mathbf{Z}P/\langle \rho, \sigma \rangle \oplus \mathbf{Z}P/\langle \tau, \rho \rangle \oplus \mathbf{Z}P/\langle \sigma, \tau \rangle \oplus \mathbf{Z}P/\langle \rho\sigma, \rho\tau \rangle \to \mathbf{Z} \to 0.$

We have only to show for these cases that L° is not a direct summand of any quasi-permutation P-module. For brevity, a direct summand of a quasi-permutation module is called a *d-quasi-permutation* module.

It should be noted that the case (i) was examined by Miyata (unpublished, 1974) and Hürlimann [5]. We give here a proof of this case for completeness. Let P be as in the case (i) and let I be the augmentation ideal of $\mathbf{Z}P$. Define the homomorphism $\phi : \mathbf{Z}P/\langle \sigma \rangle \oplus \mathbf{Z}P/\langle \tau \rangle \oplus \mathbf{Z}P/\langle \sigma\tau \rangle \to \mathbf{Z}P$ as

$$\phi((1,0,0)) = 1 + \sigma, \quad \phi((0,1,0)) = 1 + \tau, \quad \phi((0,0,1)) = 1 + \sigma\tau.$$

Restricting ϕ to L, we obtain the epimorphism $\tilde{\phi} : L \to I$. It is easy to see that $\mathrm{Ker}\, \tilde{\phi} \cong \mathbf{Z}^2$ is a trivial P-module. Hence we have the exact sequence

$$0 \to \mathbf{Z}^2 \to L \to I \to 0.$$

However, it is known [1, 3] that I° is not a d-quasi-permutation module. This concludes that L° is not a d-quasi-permutation module.

Next, consider the case (ii), (a). We restrict the exact sequence to the subgroup $P' = \langle \rho\sigma, \rho\tau \rangle$ of P. Then we see that $\mathbf{Z}P/\langle \rho, \sigma \rangle \oplus \mathbf{Z}P/\langle \rho, \tau \rangle \oplus \mathbf{Z}P/\langle \sigma, \tau \rangle \cong \mathbf{Z}P'/\langle \rho\sigma \rangle \oplus \mathbf{Z}P'/\langle \rho\tau \rangle \oplus \mathbf{Z}P'/\langle \sigma\tau \rangle$ as P'-modules. This shows by the case (i) that L° is not a d-quasi-permutation P'-module. Thus L° is also not a d-quasi-permutation P-module.

The case (ii), (b) is slightly complicated. We set $M = \mathbf{Z}P/\langle \rho \rangle \oplus \mathbf{Z}P/\langle \sigma \rangle \oplus \mathbf{Z}P/\langle \tau \rangle \oplus \mathbf{Z}P/\langle \rho\sigma \rangle \oplus \mathbf{Z}P/\langle \sigma\tau \rangle \oplus \mathbf{Z}P/\langle \tau\rho \rangle \oplus Z^3$ and $N = \mathbf{Z}P/\langle \rho, \sigma \rangle \oplus \mathbf{Z}P/\langle \tau, \rho \rangle \oplus \mathbf{Z}P/\langle \sigma, \tau \rangle \oplus \mathbf{Z}P/\langle \rho\sigma, \rho\tau \rangle$. Define the 9 homomorphisms as

$$\mathbf{Z}P/\langle \rho \rangle \to N \qquad \text{by } 1 \to (1, -1, 0, 0)$$

$$\mathbf{Z}P/\langle \sigma \rangle \to N \qquad \text{by } 1 \to (1, 0, -1, 0)$$

$$\begin{array}{ll} \mathbf{Z}P/\langle \tau \rangle \to N & \text{by } 1 \to (0,1,-1,0) \\ \mathbf{Z}P/\langle \rho\sigma \rangle \to N & \text{by } 1 \to (1,0,0,-1) \\ \mathbf{Z}P/\langle \sigma\tau \rangle \to N & \text{by } 1 \to (0,0,1,-1) \\ \mathbf{Z}P/\langle \tau\rho \rangle \to N & \text{by } 1 \to (0,1,0,-1) \\ \mathbf{Z} \to N & \text{by } 1 \to (1+\tau, -(1+\sigma), 0, 0) \\ \mathbf{Z} \to N & \text{by } 1 \to (1+\tau, 0, -(1+\rho), 0) \\ \mathbf{Z} \to N & \text{by } 1 \to (1+\tau, 0, 0, -(1+\sigma\tau)). \end{array}$$

By adding these homomorphisms, we obtain the homomorphism $\Psi : M \to N$. It is easy to see that the image of Ψ coinsides with L. Set $H = \operatorname{Ker} \Psi$. Then we have the exact sequence

$$0 \to H \to M \to L \to 0.$$

By direct computation we show that $M^{P'} \to L^{P'}$ is surjective for each subgroup P' of P. Therefore it follows that $H^1(P', H) = 0$ for each subgroup P' of P, where $H^i(\ ,\ )$ denotes the Tate cohomology group.

Assume that L° is a d-quasi-permutation. Then, by (the proof of) [3, 1.6], there is an exact sequence

$$0 \to T \to D \to L \to 0,$$

where T is a permutation module and D is a direct summand of permutation module. Taking the pullback of $M \to L \leftarrow D$, we obtain $H \oplus D \cong T \oplus M$. Since P is a 2-group, H is locally isomorphic to a permutation module S. We denote the completion of a module V at 2 by V^*. Then we have $H^* \cong S^*$. Set $\lambda = \rho\sigma\tau$.

Considering $\langle \lambda \rangle$-invariants we have $(L^*)^{\langle \lambda \rangle} \cong (\mathbf{Z}^*)^3$ and

$$(H^*)^{\langle \lambda \rangle} \cong (\mathbf{Z}^*P/\langle \rho\sigma, \tau \rangle \oplus \mathbf{Z}^*P/\langle \tau\rho, \sigma \rangle \oplus \mathbf{Z}^*P/\langle \sigma\tau, \rho \rangle)^2.$$

Since $\operatorname{rank}_Z H = 20$ and $\operatorname{rank}_Z H^P = 6$, we have

$$H^* \cong Z^*P/P_1 \oplus \mathbf{Z}^*P/P_2 \oplus \mathbf{Z}^*P/P_3 \oplus \mathbf{Z}^*P/P_4 \oplus \mathbf{Z}^*P/P_1' \oplus \mathbf{Z}^*P/P_2',$$

where $|P_1| = |P_2| = |P_3| = |P_4| = 2$ and $|P_1'| = |P_2'| = 4$. Using the cohomology sequence of the exact sequence $0 \to L \to N \to \mathbf{Z} \to 0$, we see that $H^1(P, L) \cong \mathbf{Z}/2\mathbf{Z}$ and $H^0(P, L) \cong (\mathbf{Z}/4\mathbf{Z})^3$. Further, taking the cohomology sequence of the exact sequence $0 \to H \to M \to L \to 0$, we have the exact sequence $H^0(P, H) \to H^0(P, M) \to H^0(P, L) \to H^1(P, H) = 0$. By direct computation we also have $H^0(P, H) \cong (\mathbf{Z}/2\mathbf{Z})^4 \oplus (\mathbf{Z}/4\mathbf{Z})^2$ and $H^0(P, M) \cong (\mathbf{Z}/2\mathbf{Z})^4 \oplus (\mathbf{Z}/8\mathbf{Z})^3$. Therefore we obtain the exact sequence

$$(\mathbf{Z}/2\mathbf{Z})^4 \oplus (\mathbf{Z}/4\mathbf{Z})^2 \to (\mathbf{Z}/2\mathbf{Z})^6 \oplus (\mathbf{Z}/8\mathbf{Z})^3 \to (\mathbf{Z}/4\mathbf{Z})^3 \to 0.$$

However, it is easy to see that the kernel of $(\mathbf{Z}/2\mathbf{Z})^6 \oplus (\mathbf{Z}/8\mathbf{Z})^3 \to (\mathbf{Z}/4\mathbf{Z})^3$

is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^9$. This is a contradiction, because $(\mathbf{Z}/2\mathbf{Z})^4 \oplus (\mathbf{Z}/4\mathbf{Z})^2$ never maps onto $(\mathbf{Z}/2\mathbf{Z})^9$. Thus L° is not a d-quasi-permutation.

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Finally consider the case where $p \neq 2$ and $r \geqq 2$. We now prove that L° is not a d-quasi-permutation. Restricting the exact sequence $(*)$ to the subgroup $P' = P_1 \cap P_2$, regarding P/P' as P and using the lemma as in the case where $p = 2$, we can reduce this case to the one where $|P| = p^2$. From now on, assume that P is of order p^2. Under this assumption it is noted that there exist exactly $p+1$ subgroups of order p in P. Let P_i, $1 \leqq i \leqq r$, be distinct subgroups of order p in P and set $N = \bigoplus_{i=1}^r \mathbf{Z}P/P_i$. Let $\varepsilon_i : \mathbf{Z}P/P_i \to \mathbf{Z}$, $1 \leqq i \leqq r$, be the augmentation maps and let

$$\Phi = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r) : N \to \mathbf{Z}.$$

Set $L = \operatorname{Ker} \Phi$. Then we have the exact sequence

$$0 \to L \to N \to \mathbf{Z} \to 0.$$

For $1 \leqq i \leqq r$, set $P/P_i = \langle \rho_i \rangle$ and $s_i = \sum_{j=0}^{p-1} \rho_i^j \in \mathbf{Z}P/P_i$. Define the homomorphisms as

$$\begin{array}{ll}
\mathbf{Z}P \to N & \text{by } 1 \to (1, -1, 0, \dots, 0) \\
\mathbf{Z}P \to N & \text{by } 1 \to (1, 0, -1, 0, \dots, 0) \\
 & \cdots \\
 & \cdots \\
\mathbf{Z}P \to N & \text{by } 1 \to (1, 0, \dots, 0, -1) \\
\mathbf{Z} \to N & \text{by } 1 \to (s_1, -s_2, 0, \dots, 0) \\
\mathbf{Z} \to N & \text{by } 1 \to (s_1, 0, -s_3, 0, \dots, 0) \\
 & \cdots \\
 & \cdots \\
\mathbf{Z} \to N & \text{by } 1 \to (s_1, 0, 0, \dots, 0, -s_r) \\
\mathbf{Z}P/P_1 \to N & \text{by } 1 \to (\rho_1 - 1, 0, \dots, 0) \\
\mathbf{Z}P/P_2 \to N & \text{by } 1 \to (0, \rho_2 - 1, 0, \dots, 0) \\
 & \cdots \\
 & \cdots \\
\mathbf{Z}P/P_r \to N & \text{by } 1 \to (0, 0, \dots, 0, \rho_r - 1).
\end{array}$$

Set $M = \mathbf{Z}P^{r-1} \oplus \mathbf{Z}^{r-1} \oplus \bigoplus_{i=1}^r \mathbf{Z}P/P_i$ and define the homomorphism $\Psi : M \to N$ as the sum of the above homomorphisms. Then it is easy to see that $\operatorname{Im}\Psi = L$. Further set $H = \operatorname{Ker}\Psi$. Then we have the exact sequence

$$0 \to H \to M \to L \to 0.$$

As is easily seen, we have

$$\begin{aligned} &\operatorname{rank}_Z N = pr, && \operatorname{rank}_Z N^P = r,\\ &\operatorname{rank}_Z L = pr - 1, && \operatorname{rank}_Z L^P = r - 1,\\ &\operatorname{rank}_Z M = \left(p^2 + 1\right)(r - 1) + pr, && \operatorname{rank}_Z M^P = 3r - 2,\\ &\operatorname{rank}_Z H = p^2(r - 1) + r, && \operatorname{rank}_Z H^P = 2r - 1. \end{aligned}$$

By direct computation we can show that $M^{P'} \to L^{P'}$ is surjective for each subgroup P' of P. Hence it follows that $H^1(P', H) = 0$ for each subgroup P' of P.

Assume that L° is a d-quasi-permutation. Then, as in the proof of the case where $p = 2$, it follows that H is a direct summand of a permutation module. Since P is a p-group, H is locally isomorphic to a permutation module S.

We denote the completion of a module V at p by V^*. Then we can decompose $H^* \cong S^* \cong (\mathbf{Z}^*P)^s \oplus (S')^*$, where S' does not have $\mathbf{Z}P$ as a direct summand. It is easy to see that $\operatorname{rank}_Z S' = p^2(r - s - 1) + r$ and $\operatorname{rank}_Z (S')^P = 2r - s - 1$. Since each indecomposable component of S' is of rank 1 or p over $\mathbf{Z}$, we have $\operatorname{rank}_Z S' \leqq p(2r - s - 1)$. If $r - s - 3 \geqq 0$, it follows from the fact that $r \leqq p + 1 < 2p$ that $\operatorname{rank}_Z S' > p(2r - s - 1)$, which is a contradiction. Hence we have $r - s - 2 \leqq 0$, that is, $r - 2 \leqq s$. Since $s \leqq r - 1$, this shows that $s = r - 2$ or $r - 1$.

Forming the pushout of $(S')^* \leftarrow S^* \to M^*$, we obtain the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & (\mathbf{Z}^*P)^s \arrow[r, equal] \arrow[d] & (\mathbf{Z}^*P)^s \arrow[d] & & \\
0 \arrow[r] & S^* \arrow[r] \arrow[d] & M^* \arrow[r] \arrow[d] & L^* \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & (S')^* \arrow[r] \arrow[d] & (M')^* \arrow[r] \arrow[d] & L^* \arrow[r] & 0 \\
 & 0 & 0. & &
\end{tikzcd}$$

Then the columns are split, and so $(M')^* = (\mathbf{Z}^*P)^{r-s-1} \oplus (\mathbf{Z}^*)^{r-1} \oplus \bigoplus_{i=1}^{r} \mathbf{Z}^*P/P_i$. Set $U^* = \sum_{P'}(M^*)^{P'}$ where P' runs over all subgroups of order p in P. Since $(\mathbf{Z}^*)^{r-1} \oplus \bigoplus_{i=1}^{r} \mathbf{Z}^*P/P_i \subseteq U^*$, there exists an epimorphism $(\mathbf{Z}^*P)^{r-s-1} \to M^*/U^* \to L^*/\Psi^*(U^*)$. It can be shown that $\Psi*(U^*) = Z^*P(\rho_1 - 1, 0, \dots, 0) + \mathbf{Z}^*P(0, \rho_2 - 1, 0, \dots, 0) + \cdots + \mathbf{Z}^*P(0, \dots, 0, \rho_r - 1) + \mathbf{Z}^*P(p, -p, 0, \dots, 0) + \mathbf{Z}^*P(p, 0, -p, 0, \dots, 0) + \cdots + \mathbf{Z}^*P(p, 0, \dots, 0, -p)$. Hence it follows that $L^*/\Psi^*(U^*) \cong \mathbf{Z}/p\mathbf{Z}(1, -1, 0, \dots, 0) + \mathbf{Z}/p\mathbf{Z}(1, 0, -1, 0, \dots, 0) + \cdots + \mathbf{Z}/p\mathbf{Z}(1, 0, \dots, 0, -1) \cong (\mathbf{Z}/p\mathbf{Z})^{r-1}$. Therefore we obtain an epimorphism $(\mathbf{Z}^*P)^{r-s-1} \to (\mathbf{Z}/p\mathbf{Z})^{r-1}$. However, $s = r - 2$ or $r - 1$, as seen above. This shows that $r = 2$ and $s = 0$. Then we have $\operatorname{rank}_Z H = p^2 + 2$ and $\operatorname{rank}_Z H^P = 3$. This is a contradiction, because $H^*(\cong (S')^*)$ does not have $\mathbf{Z}^*P$ as a direct summand. Thus the proof is complete.

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